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A MATHEMATICAL RECREATION.

Some years ago The Mathematics Teacher described the work an attorney in Washington had done in relation to the right triangle, in which he was interested as a recreation.

The following communication is also from an attorney, Mr. Alfred L. Booth, of Provo, Utah. It outlines some investigations he has made, not as a mathematician, but as one interested in numbers. Such recreational use of the simple facts of mathematics seems to emphasize the fascination of the subject, even to those who are not professionally engaged in its study.

As a matter of interest, the editor has appended a note showing a technical method of investigating the general case of one phase of this same subject. Others may wish to make similar investigations of other parts of it.

SOME ANGLES OF THE RIGHT TRIANGLE.

By Alfred L. Booth.

In attempting to determine: (1) Can a right triangle be geometrically constructed, having one of its angles exactly one degree? and (2) How near to an isosceles commensurable right triangle can a construction approach? the following methods were pursued.

1. Beginning with that most familiar theorem in geometry: "The sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse," and taking a triangle with sides 3, 4 and 5, it will be observed that the square of the shorter leg 3 (9) is equal to the sum of the other leg and the hypotenuse (4 and 5).

The square of any odd number may in like manner be divided into two parts differing by unity. The odd number and the two parts of its square will always form the lengths of the sides of a right triangle.

As illustrations:

$$3^2 = 9 = 4 + 5;\quad 3^2 + 4^2 = 5^2.$$

$$5^2 = 25 = 12 + 13;\quad 5^2 + 12^2 = 13^2.$$

$$7^2 = 49 = 24 + 25;\quad 7^2 + 24^2 = 25^2.$$

$$9^2 = 81 = 40 + 41;\quad 9^2 + 40^2 = 41^2.$$

And so on indefinitely.

Now $113^2 = 12769 = 6384 + 6385$.

The smallest angle of a right triangle formed by lines of 113, 6,384 and 6,385 is 1° 00′ 51″.

Again; $115^2 = 13{,}225 = 6{,}612 + 6{,}613$.

The smallest angle of the right triangle with sides 115, 6,612 and 6,613 is 0° 59′ 48″.

In the one case the smallest angle is slightly greater, and in the other it is slightly less than one degree; in neither case however is it exactly 1°.

The principle can be carried on to infinity.

There being no odd number between 113 the number which gives an angle a little above 1°, and 115 which gives an angle just under 1°, the effort along that line was discontinued. In all the cases given, the sides of the respective triangles are always commensurable, and the triangles must of necessity contain in each instance a right angle.

2. This method could not be pursued in the opposite direction, that is, in trying to construct a right triangle whose legs are more nearly equal than 3 and 4. This follows from the fact that the only numbers less than 5 are 1, 2, 3 and 4 and there are no two of these except 3 and 4, the sum of whose squares will form a perfect square.

Partially by accident, and partly by trial, combined, the following sets of numbers were discovered.

A casual inspection reveals no special relationship between these numbers. It will be found by trial, however, that in each set the square of the number in column A is equal to the sum of the squares of the corresponding numbers in columns B and C. The numbers in the respective sets of Columns B and C differ by unity, and therefore if a right triangle is formed with the numbers in Set III (29, 21 and 20) the two acute angles will be more nearly equal than they will in set II (5, 4, 3) and so in any set succeeding III the numbers being larger and differing in each case by unity, will give a right triangle approximating the isosceles, yet never reaching it exactly.

	A	B	C
I	1		
II	5	4	3
III	29	21	20
IV	169	120	119
V	985	697	696
VI	5741	4060	4059
VII	33461	23661	23660
VIII	195025	137904	137903
IX	1136689	803761	803760
X	6625109	4684660	4684659
XI	38613965	27304197	27304196
XII	225058681	159140520	199140519
XIII	1311738121	927538921	927538920

Each number in Column A is found by multiplying the preceding number by six, and subtracting from the result the number preceding the one multiplied.

Again illustrating: 5 (the number opposite II in Column A) multiplied by six equals 30; subtract 1 (the number opposite I in Column A) gives 29 the next number (III) in Column A.

6 × 29 (III Column A) equals 174; Substract 5 (II, Column A) equals 169 (IV Column A).

6 × 169 minus 29 equals 985 (V Column A).

6 × 985 minus 169 equals 5,741 (VI Column A).

And so on to any desired number of results.

Going to columns B and C.—If the sum of a pair in these columns is multiplied by 4, and the number in Column A preceding this pair is added, the result gives the next number in Column A.

Illustrations: 4 plus 3 (II Columns B and C) equals 7. Four times 7 equals 28, to which add 1 (I Column A) gives 29 (III Column A). Twenty-one plus 20 (III Columns B and C) equals 41; this multiplied by 4 gives 164; add 5 (II Column A) gives 169 (IV Column 4). And so on.

This process has been carried on until the following three numbers were obtained (A) 399133058537705128729; (B) 282229692287738524680; (C) 282229692287738524679. The square of (A) equals the sum of the squares of (B) and (C).

None of the triangles formed with any of the sets of figures was such that the difference between the two acute angles was either one degree, or any multiple or fraction out of which one degree could be formed; hence the first question at the beginning of this article to date is answered in the negative; to the second question the answer is infinity.

A practical application may be made of the principle where it is desired to construct a right triangle without instruments other than a tape; or where one wishes to test the accuracy of a purported right angle. Because with carefully measured lines of a triangle with either 29, 21 and 20; 169, 120 and 119; 985, 697 and 696, one of the angles must be an exact right angle.

Going to the speculative: A right triangle having the sides before given of 399133058537705128729 for the hypotenuse, and 282229692287738524680 and 282229692287738524679 for the legs, supposing the hypotenuse to be the diameter of the earth's orbit around the sun, will be so nearly isosceles that the lengths of the legs will differ by less than the one-thousandth of a hairsbreadth. And the sine of each acute angle will coincide with the sine of an angle of 45° (or the square root of 0.500000000000000) to twenty figures. By increasing the numbers, the identity can be extended to any desired length.

PROVO, UTAH.

Note.—To find the formulas for the sides of all right triangles such that the hypotenuse and one leg of each have a given constant difference. Let the legs be x and y, the hypotenuse $x + c$; then $(x + c)^2 = x^2 + y^2$ and $y^2 = 2cx + c^2$.

But $2cx + c^2$ will be a perfect square if it takes the form $2c(2cn + 2cn^2) + c^2$, whatever the value of n.

$$\therefore y^2 = 4c^2n^2 + 4c^2n + c^2 = c^2\,(2n + 1)^2.$$

From which

$$y = c(2n + 1)$$

$$x = 2cn(n + 1)$$

and

$$x + c = c(2n^2 + 2n + 1).$$

If n takes all integral values a complete set of right triangles fulfilling the conditions will be obtained.

For the case $c = 1$, the results will be the same as those obtained in Mr. Booth's first table. It is evident from the formulas that the sides of the triangles whose sides have any other difference can be obtained by multiplying the numbers in this set by the required difference.

When $c = 1$, the resulting formulas also show the facts used by Mr. Booth in his work; for y has the general form for an odd number, $2n + 1$, and its square, $4n^2 + 4n + 1$, (y^2), is composed of two integers differing by one, $2n^2 + 2n$, (x) and $2n^2 + 2n + 1$, $(x + 1)$.

Also

$$(2n^2 + 2n)^2 + (4n^2 + 4n + 1) = (2n^2 + 2n + 1)^2$$

or

$$x^2 + y^2 = (x + 1)^2.$$

E. R. S.